$$\therefore \frac{v^2x^3}{x^3-r^3}=2f(h+d)=\frac{2(1-\delta)\ (h+d)g}{1+\delta}$$

$d$ can be found by either method in Vol. I, page 134.

**202. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.**

Three equal, uniform, similar rods $AB$, $BC$, $CD$, freely jointed at $B$ and $C$, are hung from a point by two equal strings attached at $A$ and $D$. Find the position of equilibrium.

**Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.**

By symmetry, the strings, of length $l$, say, make equal angles with the vertical, as do also the rods $AB$ and $DC$; denote these angles by $\theta$ and $\phi$, respectively. The rod $BC$ is horizontal. Denote the length of each rod by $a$, the weight by $w$, and the depth of the center of gravity of the system below the point of support by $z$, the strings being regarded as weightless.

$$z=[w(l\cos\theta+\tfrac{1}{2}a\cos\phi)+w(l\cos\theta+a\cos\phi)+w(l\cos\theta+\tfrac{1}{2}a\cos\phi)]/3w$$
$$=\tfrac{1}{3}[3l\cos\theta+2a\cos\phi].$$

For equilibrium, the value of $z$ must be a maximum.

$$\therefore 0=3l\sin\theta\, d\theta+2a\sin\phi\, d\phi \dots (1).$$

Also, by horizontal projection,

$$a=2l\sin\theta+2a\sin\phi\dots(2).$$
$$\therefore 0=l\cos\theta\, d\theta+a\cos\phi\, d\phi\dots(3).$$

$\therefore 3\tan\theta=2\tan\phi$ (by eliminating $d\theta$ and $d\phi$ from (1) and (2)). This equation, with equation (3), gives the position of equilibrium.

Also solved by G. B. M. Zerr and J. Scheffer.

## AVERAGE AND PROBABILITY.

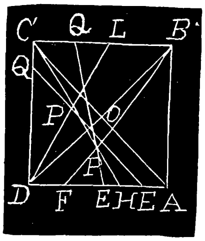
**187. Proposed by HENRY HEATON, Belfield, N. D.**

Through every point of a given square straight lines are drawn in every possible direction, terminating in the sides of the square. What is the average length of such lines?

**II. Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.**

Let $ABCD$ be the given square, side $a$. $P$ the randon point coordinates $(u, v)$. On account of the symmetry of the square we will confine $P$ to the triangle $ADC$. Let $EQ$ be the random line through $P$, $m=\tan\theta=\tan QED$. For the area $AOD$, $E$ must fall on $HF$ to intersect opposite sides and on $AH$ to intersect adjacent sides. For the area $COD$, $E$ must fall on

$HD$ to intersect opposite sides and on $AH$ to intersect adjacent sides. For opposite sides $EQ=a\operatorname{cosec}\theta$. For adjacent sides $EQ=(a-u)\sec\theta+v\operatorname{cosec}\theta$. For opposite sides, the limits for $AOD$ are, of $v$, 0 and $\frac{1}{2}a$; of $u$, $(a-v)$ and




$v$; of $\theta$, $-\tan^{-1}\left(\frac{a-v}{u}\right)=\theta_2$ and $\tan^{-1}\left(\frac{a-v}{a-u}\right)=\theta_1$; for $COD$, the limits of $u$ are $\frac{1}{2}a$ and $a$; of $v$, $u$ and $(a-u)$; of $\theta$, $-\tan^{-1}\left(\frac{v}{a-u}\right)=\theta_3$ and $\theta_1$. For adjacent sides the the limits of $u$ are 0 and $a$, of $v$, 0 and $u$, of $\theta$, $\tan^{-1}\frac{v}{u}=\theta_4$ and $\theta_1$. Let $\triangle$ be the required average length. Then we have for the denominator:

$$D=\int_0^{\frac{1}{2}a}\int_v^{a-v}\int_{\theta_2}^{\theta_1} dv\,du\,d\theta+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\int_{\theta_3}^{\theta_1} du\,dv\,d\theta+\int_0^a\int_0^u\int_{\theta_1}^{\theta_4} du\,dv\,d\theta$$

$$=\int_0^{\frac{1}{2}a}\int_v^{a-v}\left[\tan^{-1}\left(\frac{a-v}{a-u}\right)+\tan^{-1}\left(\frac{a-v}{u}\right)\right]dv\,du$$

$$+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\left[\tan^{-1}\left(\frac{a-v}{a-u}\right)+\tan^{-1}\left(\frac{v}{a-u}\right)\right]du\,dv$$

$$+\int_0^a\int_0^u\left[\tan^{-1}\left(\frac{v}{u}\right)-\tan^{-1}\left(\frac{a-v}{a-u}\right)\right]du\,dv$$

$$=\int_0^{\frac{1}{2}a}\left[2(a-v)\log\frac{(a-v)\sqrt{2}}{\sqrt{[(a-v)^2+v^2]}}+\frac{\pi}{\frac{1}{2}}(a-v)-2v\tan^{-1}\left(\frac{a-v}{v}\right)\right]dv$$

$$+\int_{\frac{1}{2}a}^{a}\left[2(a-u)\log\frac{(a-u)\sqrt{2}}{\sqrt{[(a-u)^2+u^2]}}-\frac{\pi}{2}(a-u)+2u\tan^{-1}\left(\frac{u}{a-u}\right)\right]du$$

$$+\int_0^a\left[(a-u)\log\frac{\sqrt{[(a-u)^2+a^2]}}{(a-u)\sqrt{2}}-u\log\sqrt{2}+\frac{\pi a}{4}-a\tan^{-1}\left(\frac{a}{a-u}\right)\right]du$$

$$=\tfrac{1}{4}a^2(\pi+2\log 2).$$

For the numerator we have

$$N=\int_0^{\frac{1}{2}a}\int_v^{a-v}\int_{\theta_2}^{\theta_1} a\operatorname{cosec}\theta\,dv\,du\,d\theta+\int_{\frac{1}{2}a}^{a}\int_{a-u}^{u}\int_{\theta_3}^{\theta_1} a\operatorname{cosec}\theta\,du\,dv\,d\theta$$

$$+\int_0^a\int_0^u\int_{\theta_1}^{\theta_4}[(a-u)\sec\theta+v\operatorname{cosec}\theta]du\,dv\,d\theta$$

$$=a\int_0^{\frac{1}{2}a}\int_v^{a-v}\log\left[\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a-u)}{\sqrt{[u^2+(a-u)^2]}+u}\right]dv\,du$$

$$+a\int_{\frac{1}{2}a}^{a}\int_{a-v}^{u}\log\left[\left(\frac{v}{a-v}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a-u)}{\sqrt{[(a-u)^2+v^2]}+a-u}\right)\right]du\,dv$$

$$-\int_0^a\int_0^u\left[(a-u)\log\left(\frac{u}{a-u}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}+(a-v)}{\sqrt{[u^2+v^2]}+v}\right)\right]$$

$$+v\log\left[\left(\frac{v}{a-v}\right)\left(\frac{\sqrt{[(a-u)^2+(a-v)^2]}-(a-u)}{\sqrt{[u^2+v^2]}-u}\right)\right]du\,dv$$

$$=2a\int_0^{\frac{1}{2}a}\left[v\log\left(\frac{\sqrt{[(a-v)^2+v^2]}+v}{a-v}\right)-(a-v)\log(\sqrt{2}+1)+(a-v)\sqrt{2}\right.$$

$$\left.-\sqrt{[(a-v)^2+v^2]}\right]dv$$

$$+2a\int_{\frac{1}{2}a}^{a}\left[2(a-u)\log(\sqrt{2}+1)-(a-u)\log\{\sqrt{[(a-u)^2+u^2]}+u\}\right.$$

$$\left.-u\log\left(\frac{\sqrt{[(a-u)^2+u^2]}+(a-u)}{u}\right)+(a-u)\log)a-u)\right]du$$

$$+\tfrac{1}{2}\int_0^a\left[2au-u^2-a^2\sqrt{2}+(a-u)\sqrt{[(a-u)^2+a^2]}\right.$$

$$\left.-a^2\log\left(\frac{\sqrt{[(a-u)^2+a^2]}-(a-u)}{a(\sqrt{2}-1)}\right)\right]du$$

$$=\frac{a^3}{12}[9(\sqrt{2}+1)\log(\sqrt{2}+1)-4-5\sqrt{2}].$$

$$\therefore\ \Delta=\frac{N}{D}=\frac{a}{3}\left(\frac{9(\sqrt{2}+1)\log(\sqrt{2}+1)-4-5\sqrt{2}}{\pi+2\log 2}\right)$$

NOTE.—This solution differs from Mr. Heaton's in that Dr. Zerr assumes the lines to be secant lines, that is, lines whose extremities lie on the sides of the square. Mr. Heaton intended the lines to be radial lines, that is, lines the extremities of which are the random point and points on the sides of the square. In both solutions, the same law of distribution has been assumed. ED. F.

188. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Find the average length of a hole at random through a given (a) sphere, (b) cube.

Solution by G. B. M., ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

(a) Let one point remain fixed and be the origin. Then $x^2+y^2+z^2=2ax=d^2$ is the equation to the sphere of radius $a$.